

Violent Relaxation, Phase Mixing, and Gravitational Landau Damping

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Abstract

This paper outlines a geometric interpretation of flows generated by the collisionless Boltzmann equation, focusing in particular on the coarse-grained approach towards a time-independent equilibrium. The starting point is the recognition that the collisionless Boltzmann equation is a noncanonical Hamiltonian system with the distribution function f as the fundamental dynamical variable, the mean field energy $\mathcal{H}[f]$ playing the role of the Hamiltonian and the natural arena of physics being Γ , the infinite-dimensional phase space of distribution functions. Every time-independent equilibrium f_0 is an energy extremal with respect to all perturbations δf that preserve the constraints (Casimirs) associated with Liouville's Theorem. If the extremal is a local energy minimum, f_0 must be linearly stable but, if it corresponds instead to a saddle point, f_0 may be unstable. If an initial $f(t=0)$ is sufficiently close to some linearly stable lower energy f_0 , its evolution can be visualised as involving linear phase space oscillations about f_0 which, in many cases, would be expected to exhibit linear Landau damping. If instead $f(0)$ is far from any stable extremal, the flow will be more complicated but, in general, one might anticipate that the evolution can be visualised as involving nonlinear oscillations about some lower energy f_0 . In this picture, the coarse-grained approach towards equilibrium usually termed violent relaxation is interpreted as nonlinear Landau damping. Evolution of a generic initial $f(0)$ involves a coherent initial excitation $\delta f(0) \equiv f(0) - f_0$, not necessarily small, being converted into incoherent motion associated with nonlinear oscillations about some f_0 which, in general, will exhibit destructive interference. This picture allows for distinctions between regular and chaotic "orbits" in Γ : Stable extremals f_0 all have vanishing Lyapunov exponents, even though "orbits" oscillating about f_0 may well correspond to chaotic trajectories with one or more positive Lyapunov exponents.

1. Introduction and Motivation

The problem addressed in this paper is how to visualise flows generated by the collisionless Boltzmann equation (*CBE*), i.e., the gravitational analogue of the electrostatic Vlasov equation from plasma physics.

It is generally accepted that many physical problems arising in galactic dynamics and cosmology can be modeled in terms of the *CBE*, perhaps allowing also for low amplitude discreteness effects, modeled as friction and noise through the formulation of a Fokker-Planck equation, or for a coupling to a dissipative fluid described, e.g., by the Navier-Stokes equation. Astronomers recognise that an evolution described completely by the *CBE* is special because of the constraints associated with Liouville's Theorem, and that, at some level, the flow must be Hamiltonian, which precludes the possibility of any pointwise approach towards a time-independent equilibrium: in the absence of dissipation, one can only speak meaningfully of a coarse-grained approach towards equilibrium. However, there does not seem to be a clear sense of exactly how one ought to visualise a flow governed by the *CBE* or of what sort of coarse-graining one ought to implement in order to identify an approach towards equilibrium.

The conventional wisdom of galactic dynamics (cf. Binney and Tremaine 1987), as articulated, e.g., by Maoz (1991), draws sharp distinctions between different aspects of the evolution, speaking separately of phase mixing, (linear) Landau damping, and violent relaxation. However, such distinctions, even if useful in addressing specific physical effects, are arguably *ad hoc* and, as such, may obscure the overall character of the flow. Plasma physicists (cf. van Kampen 1955, Case 1959) are well acquainted with the fact that, appropriately interpreted, linear Landau damping *is* a phase mixing associated with the evolution of a wave packet constructed from a continuous set of normal modes. Moreover, even though conventional wisdom makes a sharp distinction between violent relaxation and phase mixing/Landau damping, one can argue that, as is implicit in Lynden-Bell's (1967) original paper on violent relaxation, it too is a phase mixing process.

The objective here is to present a coherent mathematical description of an evolution described by the *CBE* that manifests explicitly the Hamiltonian character of the flow. This entails a synthesis and extension of existing work in both plasma physics and galactic dynamics (cf. Morrison 1980, Morrison and Eliezur 1986, Kandrup 1989, 1998 and numerous references cited therein) which, in the context of galactic dynamics, has proven useful in understanding problems related to both linear and global stability, as well as stability in the presence of weak dissipation (cf. Kandrup 1991a,b, Perez and Aly 1996, Perez, Alimi, Aly, and Scholl 1996). Section 2 describes the precise sense in which the *CBE* is an infinite-dimensional Hamiltonian system, identifying the natural phase space, exhibiting the noncanonical Hamiltonian structure, and then speculating on the possible meaning of regular versus chaotic flows.

Section 3 turns to the problem of linear stability for time-independent equilibria. This is

addressed both in the context of the full noncanonical Hamiltonian dynamics and in terms of a simpler canonical Hamiltonian structure associated with the tangent dynamics, i.e., identifying explicitly a set of canonically conjugate variables in terms of which to analyse linear perturbations. One immediate by-product of this discussion is a simple explanation (cf. Habib, Kandrup, and Yip 1986) of linear Landau damping which manifests explicitly that it is in fact a phase mixing process: Even though a perturbation cannot “die away” in any pointwise sense, one may expect a coarse-grained approach towards equilibrium in which observables like the density perturbation $\delta\rho$ eventually decay to zero.

Section 4 generalises the preceding to the case of nonlinear stability, allowing for perturbations δf away from some equilibrium f_0 which are not necessarily small. The intuition derived from that problem is then used to motivate one possible way in which to visualise the flow associated with a generic initial $f(t=0)$. The obvious point is that a generic initial $f(0)$ can be viewed as a (possibly strongly nonlinear) perturbation of *some* equilibrium f_0 , the form of which, however, need not be known explicitly. To the extent that this interpretation is accepted, those aspects of the flow typically denoted violent relaxation should be viewed as nonlinear Landau damping/phase mixing (cf. Kandrup 1998). Section 5 concludes by describing the mathematical issues which must be resolved to make the preceding discussion rigorous and complete.

A simple mechanical model, which can help in visualising the basic ideas described in this paper, is the following: Consider a point particle moving in some complicated, many-dimensional potential $V(\mathbf{r})$ which is characterised generically by multiple extremal points but which, being bounded from below, will have a (in general nondegenerate) global minimum. If one chooses initial data corresponding to a configuration space point \mathbf{r} close to but slightly above some local minimum \mathbf{r}_0 and a velocity \mathbf{v} whose magnitude is very small, the subsequent evolution will involve linear oscillations about \mathbf{r}_0 , whether or not that point corresponds to a global minimum. The trajectory of the point particle thus corresponds to a regular orbit in what appears locally as a harmonic potential. If the initial deviation from the extremal point becomes somewhat larger, because $|\mathbf{r} - \mathbf{r}_0|$ and/or $|\mathbf{v}|$ is bigger, one would still anticipate oscillations around \mathbf{r}_0 , but these will now become nonlinear and the particle trajectory may well correspond to a chaotic orbit. Suppose, however, that \mathbf{r}_0 is *not* the global minimum. In this case, one would expect that, for initial data sufficiently far from \mathbf{r}_0 , the particle will have left the “basin of attraction” associated with the local minimum and will instead (generically) exhibit strongly nonlinear oscillations about the global minimum (it could of course oscillate around a different nonglobal minimum!). In the absence of dissipation, there is no pointwise sense in which the particle evolves towards the global minimum. However, the nonlinear oscillations in different directions will in general interfere destructively, so that any initial coherence between motions in different directions will eventually be lost (at least for times short compared with the Poincaré recurrence time). It is this loss of coherence which, for the *CBE*, gives rise to (linear or nonlinear) Landau damping.

2. The Noncanonical Hamiltonian Formulation

If one considers the Liouville equation appropriate for a collection of noninteracting particles evolving in a fixed potential $\Phi(\mathbf{x})$, the natural phase space is the six-dimensional phase space associated with the canonical pair (\mathbf{x}, \mathbf{v}) . If, however, one considers the full *CBE*, allowing for a self-consistent potential $\Phi[f(\mathbf{x}, \mathbf{v})]$ determined by the free-streaming particles, this is no longer so. In this case, the fundamental dynamical variable is the distribution function itself, and the natural phase space Γ is the infinite-dimensional phase space of distribution functions. In general, it is not easy to identify conjugate coordinates and momenta in this phase space so as to rewrite the *CBE* in the form of Hamilton's equations. However, one can still capture the Hamiltonian character at a formal algebraic level through the identification of an appropriate cosymplectic structure (cf. Arnold 1989).¹

In this context, manifesting the Hamiltonian character of the flow entails identifying a Lie bracket $[\cdot, \cdot]$, defined on pairs of phase space functionals $\mathcal{A}[f]$ and $\mathcal{B}[f]$, and a Hamiltonian functional $\mathcal{H}[f]$, so chosen that the *CBE*

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \quad (1)$$

with $\Phi(\mathbf{x}, t)$ the self-consistent potential satisfying

$$\nabla^2 \Phi = 4\pi G \rho \equiv \int d^3v f, \quad (2)$$

can be written in the form

$$\frac{\partial f}{\partial t} + [\mathcal{H}, f] = 0. \quad (3)$$

¹ One example of a noncanonical Hamiltonian system, well known to astronomers, is rigid body rotations described by the standard Euler equations (cf. Landau and Lifshitz 1960). Specifically, as described and generalised, e.g., in Kandrup (1990) and Kandrup and Morrison (1993), the Euler equations constitute a Hamiltonian system, formulated in the three-dimensional phase space coordinatised by the three components of angular momentum J_i , ($i = 1, 2, 3$), with the Hamiltonian $H[J_i] = \sum_{i=1}^3 J_i^2 / 2I_i$ (the analogue of eq. 4) defined in terms of the principal moments of inertia I_i , and the Lie bracket (the analogue of eq. 5) given as the natural bracket associated with the three-dimensional rotation group, i.e.,

$$[a, b] = \sum_{i,j,k} \epsilon_{ijk} J_k \left(\frac{\partial a}{\partial J_i} \right) \left(\frac{\partial b}{\partial J_j} \right)$$

for functions $a(J_i)$ and $b(J_i)$. As for the *CBE*, there is also a Casimir (the analogue of eq. 9), namely $C[J_i] = \sum_{i=1}^3 J_i^2$, which restricts motion to the two-dimensional constant C surface in the three-dimensional phase space.

Astronomers are also acquainted with infinite-dimensional Hamiltonian systems, at least those realisable in canonical coordinates, one simple example being the scalar wave equation $\partial_t^2 \Psi - \nabla^2 \Psi = 0$, which derives from the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int d^3x \left(\Pi^2(\mathbf{x}) + |\nabla \Psi(\mathbf{x})|^2 \right),$$

where Ψ and Π are canonically conjugate.

The Hamiltonian \mathcal{H} may be taken as

$$\mathcal{H}[f] = \frac{1}{2} \int d\Gamma v^2 f(\mathbf{x}, \mathbf{v}) - \frac{G}{2} \int d\Gamma \int d\Gamma' \frac{f(\mathbf{x}, \mathbf{v}) f(\mathbf{x}', \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|}, \quad (4)$$

with $d\Gamma \equiv d^3x d^3v$, which corresponds to the obvious mean field energy, as identified, e.g., by Lynden-Bell and Sanitt (1969). The bracket is then chosen to satisfy (Morrison 1980)

$$[\mathcal{A}, \mathcal{B}] = \int d\Gamma f \left\{ \frac{\delta \mathcal{A}}{\delta f}, \frac{\delta \mathcal{B}}{\delta f} \right\}, \quad (5)$$

where $\{g, h\}$ denotes the ordinary Poisson bracket acting on functions $g(\mathbf{x}, \mathbf{v})$ and $h(\mathbf{x}, \mathbf{v})$, and $\delta/\delta f$ denotes a functional derivative. It is straightforward to show that the operation defined by eq. (5) is a skew symmetric, bilinear form, satisfying the Jacobi identity

$$[g, [h, k]] + [h, [k, g]] + [k, [g, h]] = 0, \quad (6)$$

so that it defines a bona fide Lie bracket. However for this bracket one verifies immediately that eq. (3) reduces to the *CBE* in the form

$$\frac{\partial f}{\partial t} - \{E, f\} = 0, \quad (7)$$

where E represents the energy of a unit mass test particle, i.e.,

$$E = \frac{1}{2} v^2 + \Phi(\mathbf{x}, t). \quad (8)$$

A flow governed by the *CBE* is strongly constrained by Liouville's Theorem, which implies the existence of an infinite number of conserved quantities, the so-called Casimirs $\mathcal{C}[f]$. Specifically, the flow has the property that, for any function $\chi(f)$, the value of the phase space integral

$$\mathcal{C}[f] = \int d\Gamma \chi(f) \quad (9)$$

is invariant under time translation, i.e., $d\mathcal{C}/dt = 0$. The simplest case corresponds to the choice $\chi = f$, which leads to conservation of number (or mass):

$$\frac{d}{dt} \int d\Gamma f \equiv 0. \quad (10)$$

By analogy with finite-dimensional systems, where Noether's Theorem relates conserved quantities to continuous symmetries, these Casimirs reflect internal symmetries in the infinite-dimensional phase space Γ (Morrison and Eliezur 1986).

The Casimirs play an important role in analysing the stability of equilibrium solutions f_0 , where one must restrict attention to perturbations δf that satisfy $\delta \mathcal{C} \equiv 0$ for all possible choices of χ . As first noted by Bartholomew (1971), this demand implies that any allowed perturbation δf is related to f_0 by a canonical transformation induced by some generating function g , i.e.,

$$f \equiv f_0 + \delta f = \exp(\{g, \cdot\}) f_0. \quad (11)$$

In addition to the Casimirs, there is also at least one other conserved quantity, namely the mean field energy $\mathcal{H}[f]$. Specifically, it follows from the *CBE* that $d\mathcal{H}/dt \equiv 0$. If one considers initial data $f(0)$ characterised by a high degree of symmetry, other conserved quantities may also exist. For example, if the initial data correspond to a potential Φ which is spherically symmetric, it follows that the numerical value of the angular momentum

$$\mathbf{J} \equiv \int d^3x d^3v f \mathbf{x} \times \mathbf{v} \quad (12)$$

is necessarily conserved. However, these conserved quantities, if they exist, are on a different footing from the Casimirs since they reflect symmetries in the particle phase space, rather than internal symmetries associated with the infinite-dimensional phase space of distribution functions.

Because of the infinite number of constraints associated with the Casimirs, the evolution of f is reduced to a lower (but presumably still infinite-) dimensional phase space hypersurface, say γ . One might naively believe that, in the same sense as, e.g., for the Kortweg-de Vries equation (cf. Arnold 1989), the flow associated with the *CBE* is integrable. In point of fact, however, this is almost certainly not so (cf. Morrison 1987), the important point being that the Casimirs associated with the *CBE* are all “ultralocal” quantities which do not involve derivatives of f .

At the present time, there is no universally accepted notion of what precisely one should mean by chaos in an infinite-dimensional Hamiltonian system. However, one obvious tact entails comparing initially nearby flows and asking whether, for some given $f(t=0)$, there exist perturbations $\delta f(t=0)$ which grow exponentially. This leads naturally² to the notion of a functional Lyapunov exponent which, at least formally, can be defined by analogy with the definition of an ordinary Lyapunov exponent in a finite-dimensional system (cf. Lichtenberg and Lieberman 1992). Specifically, given the introduction of an appropriate norm $|| \cdot ||$, one can write

$$\chi = \lim_{t \rightarrow \infty} \lim_{\delta f(0) \rightarrow 0} \frac{1}{t} \frac{||\delta f(t)||}{||\delta f(0)||}. \quad (13)$$

For finite dimensional systems one knows that, independent of the choice of norm, the analogue of eq. (13) will, for a generic phase space perturbation δz , converge towards the largest Lyapunov exponent. Much less is known about the infinite-dimensional case. For specificity, it thus seems reasonable to choose $|| \cdot ||$ as corresponding to a (possibly weighted) L^2 norm defined in the phase space of distribution functions, i.e.,

$$||\delta f|| \equiv \int d\Gamma M(\mathbf{x}, \mathbf{v}) |\delta f(\mathbf{x}, \mathbf{v})|^2, \quad (14)$$

where M denotes a specified function of \mathbf{x} and \mathbf{v} . This is, e.g., the type of norm that has been used in proving theorems about linear stability.

² I thank Bruce Miller and Klaus Dietz for suggesting this point to me.

3. Linear Stability and Gravitational Landau Damping

The key fact underlying the interpretation of flows described by the *CBE* and, especially, the problem of stability, is that every time-independent equilibrium f_0 is an energy extremal with respect to “symplectic” perturbations δf of the form (11) which preserve the numerical values of every Casimir. This implies that, if one restricts attention to the reduced phase space γ obtained by freezing the value of each Casimir at its equilibrium value $C[f_0]$, every equilibrium f_0 corresponds to an isolated fixed point: To lowest order, the quantity $\delta\mathcal{H} \equiv 0$ for any symplectic δf . As explained below, if f_0 is a local energy minimum, so that, to next leading order, $\delta\mathcal{H} \geq 0$, f_0 must be linearly stable. Alternatively, if f_0 corresponds to a saddle point, so that \mathcal{H} increases for some perturbations but decreases for others, linear stability is no longer guaranteed, although one cannot necessarily infer that f_0 must be linearly unstable.

The proof that, to lowest order, $\delta\mathcal{H}$ vanishes for any perturbation of the form (11) and the computation of $\delta\mathcal{H}$ to higher order are straightforward if one expands (11) perturbatively to infer that

$$\delta f = \{g, f_0\} + \frac{1}{2}\{g, \{g, f_0\}\} + \dots \equiv \delta^{(1)}f + \delta^{(2)}f + \dots \quad (15)$$

It is easy to see that, for any $\delta^{(1)}f$, the first variation $\delta^{(1)}\mathcal{H}$ becomes

$$\delta^{(1)}\mathcal{H} = \int d\Gamma \left(\frac{1}{2}v^2 - G \int d\Gamma' \frac{f'_0}{|\mathbf{x} - \mathbf{x}'|} \right) \delta^{(1)}f = \int d\Gamma E_0 \delta^{(1)}f, \quad (16)$$

where E_0 is the particle energy associated with f_0 . However, by combining eqs. (15) and (16) and then integrating by parts, one finds that

$$\delta^{(1)}\mathcal{H} = \int d\Gamma E_0 \{g, f_0\} = - \int d\Gamma g \{E_0, f_0\} \equiv 0, \quad (17)$$

where (cf. eq. 7) the final equality follows from fact that f_0 is time-independent. Extending this calculation to one higher order shows that the second variation

$$\delta^{(2)}\mathcal{H} = -\frac{1}{2} \int d\Gamma \{g, f_0\} \{g, E_0\} - \frac{G}{2} \int d\Gamma \int d\Gamma' \frac{\{g, f_0\} \{g', f'_0\}}{|\mathbf{x} - \mathbf{x}'|}. \quad (18)$$

To help visualise what is going on, and to understand why linear stability follows if $\delta^{(2)}\mathcal{H}$ is positive for all symplectic perturbations of the form (11), suppose that, in ordinary three-dimensional space, the x - y plane corresponds to a hypersurface in the reduced γ -space of distribution functions. One can then “warp” this plane into a curved two-dimensional surface by assigning to each x - y pair a coordinate z which corresponds to the numerical value assumed by the energy \mathcal{H} . On this warped surface, the equilibrium points correspond to those pairs (x_0, y_0) which are extremal in z , so that any infinitesimally displaced point $(x_0 + \delta x, y_0 + \delta y)$ assumes a new value $z + \delta z$.

If the equilibrium point is a local energy minimum, any infinitesimal displacement on the surface necessarily increases the value of z , so that, in the neighbourhood of (x_0, y_0) , the surface has the geometry of an upward opening paraboloid. Any perturbation comes

with positive energy and corresponds to bounded motion on the paraboloid. Thus the equilibrium is linearly stable. In principle, the same conclusion also obtains if the extremal point is a local maximum, although one can show that, for realistic equilibria, $\delta^{(2)}\mathcal{H}$ is never strictly negative. If, however, the equilibrium corresponds to a saddle point, so that z increases in some directions but decreases in others, the situation becomes more complicated. In this case, the linearised dynamics implies that it is possible to combine a very large negative energy perturbation in one direction with a very large positive energy perturbation in another to generate a total perturbation with vanishing energy. In itself, this does not guarantee a linear instability, but the simple geometric argument for stability that holds for a local minimum is no longer applicable.³

That saddle points need not imply linear instability may seem surprising at first glance. However, the following two-dimensional example makes clear exactly what can go wrong:

$$H = \frac{1}{2}(v_1^2 + \omega_1^2 x_1^2) - \frac{1}{2}(v_2^2 + \omega_2^2 x_2^2). \quad (19)$$

Here $x_1 = v_1 = x_2 = v_2 = 0$ is a time-independent extremal point in the phase space which corresponds to a saddle but, nevertheless, the equilibrium is clearly stable. This model may seem somewhat contrived but, as discussed in Section V of Kandrup and Morrison (1993), such stable saddle points are not uncommon in various infinite-dimensional Hamiltonian systems.

The preceding argument for stability or lack thereof may seem somewhat unusual because it is formulated abstractly in phase space, without the introduction of conjugate coordinates and momenta. One might therefore hope that, by identifying an appropriate set of conjugate variables, a more intuitive proof could be derived. In certain cases, this is in fact possible. One knows that, when formulated in the full Γ -space, the dynamics cannot be decomposed completely into canonical variables because of the existence of the Casimirs, which correspond to null vectors of the cosymplectic structure. If, however, one passes to the reduced γ space, where the values of all the Casimirs are frozen, one might expect that, at least locally, conjugate variables do exist. Indeed, for finite-dimensional systems it follows from Darboux's Theorem (cf. Arnold 1989) that, if the cosymplectic structure has vanishing determinant, i.e., if there are no null eigenvectors, it is always possible to find a set of canonically conjugate variables, at least locally (see Section V of Kandrup and Morrison [1993] for a detailed discussion of this point).

One setting in which such a canonical formulation is possible is for the special case of linear perturbations of an equilibrium f_0 which is a function only of the one-particle energy

³ Strictly speaking, the application of this finite-dimensional argument to an infinite-dimensional Hamiltonian system requires that the reduced phase space γ be endowed with a metric, so that one knows what is meant by distance between points. In practice, this can be done by introducing an appropriate L^2 norm, which provides the natural extension of the Euclidean notion of distance to an infinite-dimensional space. In this context, a proof of stability entails showing that $\|\delta f(t)\|$ remains bounded for all times.

E , i.e., $f_0 = f_0(E)$, and for which the partial derivative $F_E \equiv \partial f / \partial E$ is strictly negative. Physically the latter restriction implies that the system does not exhibit a population inversion; mathematically it ensures that division by F_E is well defined. The basic idea, due originally to Antonov (1960), is to split the linearised perturbation δf into two pieces, δf_+ and δf_- , respectively even and odd under a velocity inversion $\mathbf{v} \rightarrow -\mathbf{v}$, and to view the single linearised perturbation equation for δf as a coupled system for δf_{\pm} .

When linearised about some equilibrium f_0 , the *CBE* reduces to

$$\partial_t \delta f - \{E, \delta f\} - \{\Phi[\delta f], f_0\} = 0, \quad (20)$$

where E is the particle energy associated with f_0 and $\Phi[\delta f]$ denotes the gravitational potential “sourced” (cf. eq. 2) by the perturbation δf . If one observes that E is an even function of \mathbf{v} , that the Poisson bracket is odd under velocity inversion, and that $\Phi[\delta f_-]$ vanishes identically, it is clear that eq. (20) is equivalent to the coupled system

$$\partial_t \delta f_+ - \{E, \delta f_+\} = 0$$

and

$$\partial_t \delta f_- - \{E, \delta f_+\} - \{\Phi[\delta f_+], f_0\} = 0. \quad (21)$$

However, if one differentiates the second of these relations with respect to t , and uses the first to eliminate $\partial_t \delta f_+$, it follows that

$$\partial_t^2 \delta f_- = \{E, \{E, \delta f_-\}\} + \{\Phi[\{E, \delta f_-\}], f_0\} \equiv F_E \mathcal{A} \delta f_-, \quad (22)$$

where \mathcal{A} denotes a linear operator. One can then show that, given the identification of δf_- and $\partial_t \delta f_-$ as conjugate variables, the equation

$$(-F_E)^{-1} \partial_t^2 \delta f_- = -\mathcal{A} \delta f_- \quad (23)$$

can be derived from the Hamiltonian

$$\begin{aligned} \hat{\mathcal{H}} &= \frac{1}{2} \int \frac{d\Gamma}{(-F_E)} (\partial_t \delta f_-)^2 + \frac{1}{2} \int d\Gamma \delta f_- \mathcal{A} \delta f_- \\ &= \frac{1}{2} \int \frac{d\Gamma}{(-F_E)} (\partial_t \delta f_-)^2 + \frac{1}{2} \int \frac{d\Gamma}{(-F_E)} \{E, \delta f_-\}^2 - \frac{G}{2} \int d\Gamma \int d\Gamma' \frac{\{E, \delta f_-\} \{E', \delta f'_-\}}{|\mathbf{x} - \mathbf{x}'|}. \end{aligned} \quad (24)$$

The connection between $\hat{\mathcal{H}}$ and the energy $\delta^{(2)}\mathcal{H}$ associated with a small symplectic perturbation is discussed in Kandrup (1989). In particular, one can show that $\hat{\mathcal{H}} > 0$ for all δf_- if and only if $\delta^{(2)}\mathcal{H} > 0$ for all symplectic perturbations.

The fact that \mathcal{A} is a symmetric (i.e., hermitian) operator facilitates a proof that the equilibrium $f_0(E)$ is linearly stable if and only if $\hat{\mathcal{H}}$ (and $\delta^{(2)}\mathcal{H}$) is positive. Specifically, a simple energy argument (cf. Laval, Mercier, and Pellat 1965) implies that the magnitude of δf_- , and hence δf , is bounded in time if \mathcal{A} is a positive operator, so that $\delta^{(2)}\mathcal{H} > 0$, whereas the possibility of perturbations with $\int d\Gamma \delta f_- \mathcal{A} \delta f_- < 0$ implies the existence of

solutions that grow exponentially. This is easy to understand in the language of normal modes. Since \mathcal{A} is symmetric, it is clear that all solutions $\delta f_- \propto \exp(st)$ have s^2 real, so that the evolution is either purely oscillatory or purely exponential. If \mathcal{A} is positive, s^2 must be negative, so that the modes are purely oscillatory. If, however, \mathcal{A} is not a positive operator, there exist modes with $s^2 > 0$, which implies an exponential instability.⁴

This sort of normal mode expansion facilitates a simple geometric picture of an infinite-dimensional configuration space of perturbations δf_- which is (locally) embeddable in the reduced γ -space. The equilibrium f_0 , which is necessarily an extremal point of the full Hamiltonian \mathcal{H} , satisfies $\delta f_- \equiv \partial_t \delta f_- \equiv 0$. An arbitrary initial perturbation entails a kinetic energy $\mathcal{K} = \int d\Gamma (-F_E)^{-1} (\partial_t \delta f_-)^2$ which is necessarily positive and a potential energy $\mathcal{W} = \int d\Gamma \delta f_- \mathcal{A} \delta f_-$ whose sign depends on the properties of \mathcal{A} . If \mathcal{A} is a positive operator, the evolution in configuration space involves a particle with “mass” $(-F_E)^{-1}$ moving in an infinite-dimensional harmonic potential which corresponds to an upwards opening paraboloid. Linear stability is therefore assured. If, however, \mathcal{A} is not always positive, $\delta f_- \equiv 0$ corresponds to a saddle point, rather than a local minimum, and the flow is linearly unstable.

In visualising all of this, there is the strong temptation to think of the normal modes as being discrete, i.e., corresponding to honest square integrable eigenfunctions rather than singular eigendistributions. This, however, is not necessarily justified.

Assuming completeness, one can always view any linear perturbation of an equilibrium $f_0(E)$ with $F_E < 0$ as a superposition of normal modes, writing δf as a formal sum

$$\delta f(\mathbf{x}, \mathbf{v}, t) = \sum_{\sigma} A_{\sigma} g_{\sigma}(\mathbf{x}, \mathbf{v}) \exp(i\sigma t), \quad (25)$$

where g_{σ} labels the eigenvector, σ is the corresponding frequency, which is necessarily real, and A_{σ} is an expansion coefficient.⁵ Modulo largely unimportant technical details, the modes then divide into two types, namely: (1) a countable set of discrete frequencies belonging to the point spectrum, for which the corresponding eigenvectors are well-behaved (e.g., square-integrable) eigenfunctions; and (2) a continuous set of frequencies belonging to the continuous spectrum, for which the eigenvectors are singular eigendistributions.

The distinction between these two types of modes is extremely important (cf. Habib, Kandrup, and Yip 1986). Because true eigenfunctions are nonsingular, they can in principle be triggered individually, i.e., one can choose a reasonable initial δf which populates only a single discrete mode. By contrast, because eigendistributions are singular, one cannot sample a single continuous mode. Rather, any smooth δf sampling the continuous spectrum must really be constructed as a wavepacket comprised of a continuous set of modes. The

⁴ In point of fact, one anticipates that, for this simple case, \mathcal{A} is guaranteed to be positive. It is believed (cf. Binney and Tremaine 1987) that any f_0 depending only on E corresponds to a spherically symmetric configuration; but assuming that the mass density ρ associated with f_0 is spherical one can prove that \mathcal{A} is indeed positive (cf. Kandrup 1989).

⁵ Strictly speaking, this sum must be interpreted (cf. Riesz and Nagy 1955) as a Stieltjes integral.

important point then is that, when evolved into the future, such a wavepacket implies a damping of coarse-grained observables like the density ρ . In other words, if the modes are continuous there is a precise sense in which the perturbation “dies away” and the system exhibits a coarse-grained approach towards the original equilibrium f_0 .

The physics here is analogous to what arises in ordinary quantum mechanics. If, in that setting, one considers a physical observable like angular momentum with a discrete spectrum, one can construct well behaved eigenstates which, when evolved into the future, maintain their coherence for all time: the only effect of the evolution is a coherently oscillating phase. If, however, one considers an observable like position or linear momentum, where the spectrum is continuous, this is no longer so. In this case, a normalisable initial state must be constructed from a continuous set of singular eigendistributions, so that the best one can do is build a localised (e.g., minimum uncertainty) wavepacket. However, when evolved into the future such a wavepacket will necessarily spread because different eigendistributions have different phase velocities.

It is this loss of coherence associated with the spreading of a wavepacket that corresponds to (linear) Landau damping. In the context of plasma physics, Landau damping was derived originally (Landau 1946) in a very different way, through the introduction of a Fourier-Laplace transform and an analysis of poles in the complex plane. However, at least for the electrostatic Vlasov equation (cf. Case 1959), i.e., the electrostatic analogue of the *CBE*, the mathematical equivalence of these two pictures of Landau damping is well understood. The physics underlying their equivalence is discussed in Kandrup (1998).

For the special case of perturbations of an homogeneous neutral plasma characterised by an isotropic distribution of velocities that is everywhere nonvanishing, the modes can be computed explicitly (cf. Case 1959), and one finds generically that

$$g_\sigma(\mathbf{x}, \mathbf{v}) = \exp(i\mathbf{k} \cdot \mathbf{x}) g_\sigma(\mathbf{v}), \quad (26)$$

where $g_\sigma(\mathbf{v})$ is a singular eigendistribution involving a Dirac delta. In this setting, an examination of the perturbation associated with a given \mathbf{k} -vector at a fixed phase space point $(\mathbf{x}_0, \mathbf{v}_0)$ yields no evidence of damping away. Rather, one finds persistent oscillations $\propto \exp[i\mathbf{k} \cdot (\mathbf{x}_0 - \mathbf{v}_0 t)]$. This is simply a manifestation of the fact that, without the introduction of some coarse-graining, one cannot speak of the system returning to equilibrium. If, however, a coarse-graining is implemented by integrating over any finite range of velocities, one discovers that the resulting $\int d^3v \delta f(\mathbf{x}, \mathbf{v}, t)$ will in fact damp away.

The obvious question, therefore, is: will perturbations δf of a generic equilibrium solution to the *CBE* correspond to discrete modes, continuous modes, or a combination of both? Unfortunately, this is a difficult question to answer. It appears impossible to calculate the modes explicitly for realistic equilibria, and a formal analysis is also difficult because the operator \mathcal{T} entering into the linearised equation $\partial_t \delta f = \mathcal{T} \delta f$ is not elliptic and involves a singular integral kernel. However, the normal modes can, and have, been computed for a variety of nontrivial equilibrium solutions to the corresponding electrostatic Vlasov equation

(cf. van Kampen 1955, Case 1959), and the results derived thereby would seem suggestive.

Perhaps the most important result derived for the Vlasov equation is that discrete modes are seemingly the exception, rather than the norm, arising only if the equilibrium in question manifests nontrivial boundary conditions, e.g., the existence of a maximum speed v_m such that $f(\mathbf{v}) \equiv 0$ for $|\mathbf{v}| > v_m$. In particular, one can prove that the modes are always purely continuous if f_0 is an analytic function of \mathbf{v} in the complex plane.⁶ The best known example of a nonempty point spectrum is the case of so-called van Kampen (1955) modes, which arise precisely in those configurations where there is maximum velocity. In the usual interpretation (cf. Stix 1962), Landau damping is understood as resulting from a resonance between “particles” (the unperturbed f_0) propagating with velocity \mathbf{v} and a “wave” (the perturbation δf) that propagates with phase velocity \mathbf{c} . Discrete van Kampen modes correspond to perturbations which propagate with a phase velocity \mathbf{c} for which $f_0(\mathbf{c}) = 0$, so that no resonance is possible.

By analogy, one might therefore conjecture (cf. Habib, Kandrup, and Yip 1986) that, for the gravitational *CBE*, most perturbations will in fact correspond to continuous modes that damp away, but that some perturbations, especially longer wavelength disturbances that probe the phase space boundaries of the system, could in fact correspond to discrete modes. In this connection, it is interesting to note that there do in fact exist exact time-dependent solutions to the *CBE*, seemingly appropriate for a system like a galaxy, that exhibit finite amplitude undamped oscillations about some time-independent f_0 (cf. Louis and Gerhart 1988, Sridhar 1989). The interesting point, then is that in all these models the time-independent f_0 contains phase space “holes,” i.e., regions in the middle of the occupied phase space region where $f_0 \rightarrow 0$. Whether these sorts of solutions are generic, and whether they could arise from reasonable initial conditions, is at the present unclear.

Finally, it should be noted that, in point of fact, one can in principle get (at least temporary) phase mixing or loss of coherence even for the much simpler case of a finite set of discrete modes. For example, if one considers the function $x(t) = \sum_{p=10}^{29} \cos(0.1pt)$ over the finite interval $0 < t < 1000$, one infers a rapid damping of the initial coherent excitation with $x = 20$ to a much smaller value oscillating about $x = 0$ with typical amplitude $|x| \sim 1$. If, however, the evolution is tracked for a somewhat longer time one finds that the initial coherence is regained. An infinite set of continuous modes differs from this toy model in two important ways, namely (1) the recurrence time is infinitely long and (2) it is impossible to consider a smooth initial excitation that does not damp.

4. Nonlinear Stability and Global Evolution

Suppose, once again, that attention is focused on some linearly stable equilibrium $f_0(E)$

⁶ The validity of Landau’s original derivation of exponential damping actually relies on the implicit assumption that f_0 is analytic. If it is not, his manipulations of contours and evaluation of poles cannot be justified.

with $F_E < 0$, but that one is now interested in the effects of larger perturbations δf , i.e., the problem of nonlinear stability. To the extent that the normal modes of the linear problem remain complete, one can still envision evolution in terms of these modes, the important point, however, being that, because of nonlinearities, the modes will now interact. This is, e.g., the basis for the standard quasilinear analyses implemented in plasma physics, which allow for the effects of the quadratic term $\nabla\Phi\cdot(\partial\delta f/\partial\mathbf{v})$ which is ignored when considering linear perturbations.

Mode-mode couplings are important in that they facilitate the transfer of energy between different modes, which makes the physics more complicated. However, one might still anticipate that, if the modes are continuous, Landau damping can and will occur. Because of the interactions between modes, the simple model of a dispersing quantum mechanical wavepacket is no longer directly applicable, but the basic phenomenon of loss of coherence is robust. Indeed, there are many examples in nonlinear dynamics of flows satisfying nonlinear evolution equations where phase mixing occurs. It thus seems reasonable to suppose that, when considering the nonlinear evolution of some perturbation δf , one will encounter nonlinear Landau damping. For the case of an electrostatic plasma, nonlinear Landau damping is a well known, and reasonably well understood, phenomenon (cf. Davidson 1972 and references cited therein). Indeed, there are simple geometries where the nonlinear evolution can be computed explicitly in the context of a systematic perturbation expansion, thus facilitating analytic formulae for exactly how this phenomenon works (cf. Montgomery 1963).

Mode-mode couplings can also lead to another important possibility, namely the onset of chaos. Because f_0 is a local energy minimum, one knows that any infinitesimal perturbation δf will simply oscillate, each eigenvector corresponding to motion in a “direction” in configuration space that is orthogonal to the motion of all the other eigenvectors. This implies that, for the fixed point f_0 , the Lyapunov exponents, which were defined in eq. (13) as probing the average linear instability of the orbit generated from some initial $f(0)$, must all vanish identically. One might anticipate further that, when evolved into the future, other phase space points sufficiently close to f_0 will also correspond to regular orbits with vanishing Lyapunov exponents. Thus, e.g., for finite-dimensional systems one knows that there is a regular phase space region of finite measure surrounding every stable periodic orbit. However, for sufficiently large δf , where mode-mode couplings become significant and the motion cannot be well approximated by orthogonal harmonic oscillations, one might anticipate that many, if not all, perturbations will evolve chaotically. If true, this would suggest that a “typical” perturbation with $\delta\mathcal{H} = \mathcal{H}[f_0 + \delta f] - \mathcal{H}[f_0]$ will evolve ergodically on (some subset of) the constant energy hypersurface in the γ -space with energy $\mathcal{H}[f_0 + \delta f]$.

This idea of the onset and development of chaos is an infinite-dimensional generalisation of what is typically found when considering the motion of a point mass in a multi-

dimensional nonlinear potential which has only one extremal point, a global minimum.⁷ Low energy orbits sufficiently close to the pit of the potential move in what is essentially a harmonic potential, so that their motion is regular. If, however, the energy is raised one finds generically that, unless the motions in different directions remain completely decoupled, there is an onset of global stochasticity which leads, for sufficiently high energies, to well developed chaotic regions.

This configuration space description is not appropriate when considering generic equilibria, where the energy $\hat{\mathcal{H}}$ associated with a small perturbation cannot be written easily as a functional of conjugate variables, and there is no guarantee that $\hat{\mathcal{H}}$ can be written as a simple sum of kinetic and potential contributions, \mathcal{K} and \mathcal{W} . Modulo technical details, one might expect that canonical phase space coordinates do exist, at least in principle, but the energy $\hat{\mathcal{H}}$ associated with the tangent dynamics could in general be an arbitrary quadratic functional $\hat{\mathcal{H}}[q, p]$ of the conjugate variables q and p . Moreover, even for the simple model of an equilibrium $f_0(E)$ with $F_E < 0$, it may not be possible to extend the canonical description to allow for arbitrarily large perturbations δf . One really needs to return to a full phase space description.

As discussed in Section 3, if for some equilibrium f_0 the second variation $\delta^{(2)}\mathcal{H}$ is positive for all δf , a linearised perturbation corresponds in phase space to stable motion on an upwards opening infinite-dimensional paraboloid. As long as this surface remains convex, one would anticipate that stability will persist and, as such, one would expect intuitively that the equilibrium could remain nonlinearly stable even for small but finite δf . The normal modes of the linearised problem become coupled, but the geometric argument for stability should remain valid. In particular, one can presumably visualise the evolution of δf as involving nonlinear *phase space* oscillations about the equilibrium point f_0 .

If, however, f_0 corresponds to a stable saddle, one might suppose that even the smallest nonlinearities could trigger an instability (cf. Moser 1968, Morrison 1987). Thus, e.g., for the simple toy model of two stable oscillators described by eq. (19), it is possible to trigger an instability by introducing even very tiny mode-mode couplings which allow energy to be transferred between modes. Indeed, as noted by Cherry (1925), if the two frequencies are in an appropriate resonance, e.g., $\omega_2^2 = 2\omega_1^2$, the introduction of a simple cubic coupling implies that initial data arbitrarily close to $x_1 = v_1 = x_2 = v_2 = 0$ can lead to solutions in which x_1 , x_2 , v_1 , and v_2 all diverge in a finite time. If true, this expectation about saddle points would suggest that, even though they can be linearly stable, they cannot represent reasonable candidate equilibria in terms of which to model real astronomical objects.

If a linearly stable f_0 corresponds to a unique extremal point in the γ -space, the surface which near f_0 is a paraboloid will remain upwards opening even if δf is very large, so that stability should persist for arbitrarily large perturbations. In other words, one would

⁷Even order truncations of the Toda potential (cf. Kandrup and Mahon 1994) provide a simple two-dimensional example.

expect that the equilibrium f_0 is globally stable: In this case, any phase space deformation δf increases the energy, and the evolution of an initial $\delta f(0)$ will involve nonlinear phase space oscillations around the unique stable fixed point.

If, however, there exist multiple extremal points in the γ -space, each corresponding to a local energy minimum, the situation is much more complicated. In this case, one would anticipate that, for sufficiently large δf , the distribution function can actually be transferred from the “basin of attraction” of one equilibrium f_0 to the “basin” of some other f_1 . In other words, the evolution of $\delta f(0)$ could yield oscillations around f_1 , rather than f_0 . By suitably fine-tuning the perturbation, one can in principle displace the system from any one basin to any other. However, by analogy with the behaviour observed in finite-dimensional systems, one might expect generically that, if the perturbation is sufficiently large, its motion can be interpreted as involving nonlinear phase space oscillations about the global energy minimum. To the extent that this is true, one would anticipate that a sufficiently large perturbation will tend generically to push f into the “basin of attraction” of the equilibrium f_0 that corresponds to a global energy minimum.

If one considers an initial perturbation $\delta f(0)$ that is sufficiently large, the subsequent evolution will in general be almost completely unrelated to the initial equilibrium f_0 and, as such, the way in which one visualises the evolution $\delta f(0)$ is really no different from the way in which one can, and arguably should, envision the evolution of a generic $f(0)$. In other words, the physical picture described above can be used equally well to visualise generic flows associated with the initial value problem, the only difference being that, in general, one may know nothing at all about what time-independent equilibria f_0 actually exist.

Specification of an initial $f(0)$ fixes the values of all the Casimirs for all times, thus determining γ , the reduced infinite-dimensional phase space which constitutes the natural arena of physics. This $f(0)$ also fixes the numerical value of the conserved energy \mathcal{H} and, as such, determines the constant energy hypersurface in the γ -space to which the flow is necessarily restricted. By analogy with finite-dimensional Hamiltonian systems (cf. Kandrup and Mahon 1994) one might expect that, when evolved into the future, $f(0)$ will exhibit a coarse-grained approach towards an invariant measure on this hypersurface, i.e., a suitably defined microcanonical distribution. If the flow associated with $f(0)$ is chaotic, one might anticipate an approach towards this invariant measure that is exponential in time. If, alternatively, the flow is regular, one might instead expect a power law approach. However, in either case one might anticipate an approach towards a “phase-mixed” invariant measure. In this context, the crucial question is then: to what extent can this invariant measure be interpreted as corresponding to a distribution function f executing phase space oscillations about one or more equilibrium solutions f_0 ?

It is easy to see that, in the γ -space, there must exist one or more extremal points with $\delta^{(1)}\mathcal{H} = 0$, these corresponding to equilibrium solutions f_0 for which all the Casimirs share the same values as the Casimirs associated with $f(0)$. Indeed, one knows that, for sufficiently smooth initial data, the *CBE* admits global existence (cf. Pfaffelmoser 1992,

Schaeffer 1991), so that δf cannot diverge and, presumably, the Hamiltonian is bounded from below. However, this implies that there must exist at least one f_0 , namely the global energy minimum (although in principle the global minimum could be degenerate). The question therefore becomes: in the basin of which f_0 (or f_0 's) does the flow reside?

In principle, the evolved distribution function f could execute phase space oscillations about any f_0 with lower energy, which one presumably depending on the initial $f(0)$. However, one might conjecture that, if the initial $f(0)$ is sufficiently far from any equilibrium f_0 , it will execute oscillations around the global minimum f_0 . The initial $f(0)$ cannot exhibit a pointwise approach towards this, or any other, f_0 . However, one might expect that, in general, the initial deviation $\delta f(0) = f(0) - f_0$ will exhibit nonlinear Landau damping so that, in terms of observables like the density ρ , δf does indeed “die away,” and one can speak of a coarse-grained approach towards the equilibrium f_0 .

5. Conclusions and Unanswered Questions

The aim of this paper is to suggest a potentially fruitful way in which to visualise flows described by the *CBE* and, in particular, the expected coarse-grained approach towards an equilibrium. No claim is made regarding mathematical rigor, and it is not clear that all the details are completely correct. However, the viewpoint developed here does have the advantage that it incorporates what *is* known rigorously about the *CBE*, and that it provides a framework in terms of which to pose precise, well defined questions. In this context, there are at least three basic questions which, if answered satisfactorily, would yield important insights into the physical properties of a flow generated by the *CBE*:

1. Will generic initial conditions exhibit effective Landau damping, thus allowing one to speak of an efficient coarse-grained evolution towards some equilibrium f_0 ? In the context of linear Landau damping, the answer to this question depends on the spectral properties of the linearised evolution equation. If the modes are all continuous, every initial perturbation will eventually phase mix away, so that physical observables like the density will damp to zero. If, however, some of the modes are discrete, it is possible to construct initial perturbations that do not damp away. At the present time, it is not clear whether, for realistic galactic models, the spectrum is purely continuous, although the investigation of various toy models is currently underway (Lynden-Bell 1997, private communication).

To the extent that N -body simulations are reliable and that, for sufficiently large N , they capture the same physics as the *CBE*, the fact that most initial conditions yield an efficient approach towards some statistical equilibrium can be interpreted as evidence that nonlinear Landau damping is in general very effective. However, there *do* exist toy models like one-dimensional gravity where one ends up with undamped oscillations. For example, the evolution of counterstreaming initial conditions in one-dimensional systems (either gravitational or electrostatic) can lead to a final state which corresponds seemingly to a distribution function f exhibiting finite amplitude undamped oscillations about a (near-

) equilibrium f_0 (cf. Mineau, Feix, and Rouet 1990). This toy model actually corroborates the physical intuition described in this paper in the sense that, as one would expect, the phase space contains a large “hole,” i.e., a region where $f_0 \rightarrow 0$. Whether or not analogous results obtain for two- and three-dimensional systems is as yet unclear, although the problem is currently under investigation (Habib, Kandrup, Pogorelov, and Ryne, work in progress).

2. Are functional Lyapunov exponents the “right” way in which to identify chaos in infinite-dimensional systems and, assuming that they are, will a generic flow associated with the *CBE* be chaotic? Given this definition, will standard results from finite-dimensional chaos remain at least approximately valid? Although not proven for generic finite-dimensional systems, there is the physical expectation that, when evolved into the future, a chaotic initial condition will evolve towards an invariant distribution on a time scale that is related somehow to the spectrum of Lyapunov exponents. This implies however that, at asymptotically late times, one can visualise the flow as densely filling a chaotic phase space region of finite measure. Assuming, however, that this is true, the Ergodic Theorem provides important information about the statistical properties of the flow, implying the equivalence of time and phase space averages (cf. Lichtenberg and Lieberman 1992).

One other point about chaos in the *CBE* should be stressed: The definition proposed in this paper is, at least superficially, completely decoupled from the (also interesting) question of whether individual orbits in a self-consistent potential generated from the *CBE* are, or are not, chaotic. This latter question refers to the behaviour of nearby trajectories in the six-dimensional particle phase space. The “natural” definition of chaos for the *CBE* should presumably reflect properties of the flow in the infinite-dimensional phase space of distribution functions.

3. For a specified initial $f(0)$, towards which equilibrium f_0 will the system evolve? Given $f(0)$, one can compute the numerical value of all possible Casimirs, thus identifying explicitly the γ -space to which the evolution is restricted. The obvious problem, then, is to identify all time-independent equilibria f_0 in γ and to determine which initial conditions correspond to which equilibria. Although unquestionably difficult, this is a problem that is both well defined mathematically and well motivated physically. Finding all equilibria is equivalent mathematically to finding all extremal points in γ . However, to the extent that one chooses to visualise the flow as involving oscillations in the γ -space, there is no question physically but that the extremal points define “basins of attraction” associated with the oscillations.

The basic points described in this paper are easily summarised:

1. The *CBE* is a Hamiltonian system, albeit an unusual one. The fundamental dynamical variable is the distribution function f , not the particle \mathbf{x} ’s and \mathbf{v} ’s; and it is not always possible (at least easily) to identify canonically conjugate variables.
2. Because the *CBE* is Hamiltonian, there can be no pointwise approach towards equilibrium. The best for which one can hope is a coarse-grained approach towards equilibrium.
3. Even though the phase space γ associated with the dynamics is infinite-dimensional, one

might expect that much of one's intuition from finite-dimensional systems remains valid. In particular, one might anticipate an asymptotic approach towards an invariant measure, and one might hope to make meaningful distinctions between regular and chaotic flows.

4. The phenomenon normally designated as linear Landau damping can be interpreted as a phase mixing of a continuous set of normal modes. Whether a small initial perturbation will always eventually Landau damp/phase mix away depends on whether the normal modes for the linearised perturbation equation are discrete or continuous.

5. To the extent that one's ordinary intuition about finite-dimensional phase spaces remains approximately valid, the evolution of generic initial data should be interpreted as involving nonlinear (phase space) oscillations about one or more energy extremals, which correspond to time-independent equilibria f_0 . The phenomenon of violent relaxation should thus be interpreted as nonlinear phase mixing/Landau damping which, if efficient, will facilitate a coarse-grained approach towards equilibrium.

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